Published for SISSA by Springer

Received: *September 23, 2014*
Accepted: *October 21, 2014*
Published: *November 5, 2014*

An octonionic formulation of the M-theory algebra

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Abstract: We give an octonionic formulation of the $\mathcal{N} = 1$ supersymmetry algebra in $D = 11$, including all brane charges. We write this in terms of a novel outer product, which takes a pair of elements of the division algebra $\mathbb{A}$ and returns a real linear operator on $\mathbb{A}$. More generally, with this product comes the power to rewrite any linear operation on $\mathbb{R}^n$ $(n = 1, 2, 4, 8)$ in terms of multiplication in the n-dimensional division algebra $\mathbb{A}$. Finally, we consider the reinterpretation of the $D = 11$ supersymmetry algebra as an octonionic algebra in $D = 4$ and the truncation to division subalgebras.

Keywords: Extended Supersymmetry, M-Theory

ArXiv ePrint: 1402.4649

doi:10.1007/JHEP11(2014)022

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Contents

1 Introduction

A recurring theme in the study of supersymmetry and string theory is the connection to the four division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. See, for example, [1–9]. The octonions are of particular interest in this context since they may be used to describe representations of the Lorentz group in spacetime dimensions $D = 10, 11$, where string and M-theory live. Furthermore, the octonions provide a natural explanation [10, 11] for the appearance of exceptional groups as the U-dualities of supergravities [12] and M-theory [13, 14].

A $D = 11$ spinor with 32 components may be packaged as a 4-component octonionic column vector [6, 15]. This has prompted the question of how to write the algebra of $D = 11$ supergravity (or 'M-algebra') using octonionic supercharges Q. This was explored in [15] where the problem was highlighted that the apparently natural choice of octonionic matrices could not provide enough degrees of freedom to account for all of M-theory's brane charges. Another fundamental question that arises when writing the $\{Q, Q\}$ algebra in this way is whether or not the usual anti-commutator is really the appropriate object to study, given that the fermionic supercharges are written over a non-commutative and non-associative algebra $\mathbb{O}$.

In the present paper we tackle this problem by introducing a novel outer product, which takes a pair of elements belonging to a division algebra $\mathbb{A}$ and returns a real linear operator on $\mathbb{A}$, expressed using multiplication in $\mathbb{A}$. This product enables one to rewrite any expression involving $n\times n$ matrices and n-dimensional vectors in terms of multiplication in the n-dimensional division algebra $\mathbb{A}$. We solve the problem of the octonionic M-algebra using this product, which allows a derivation of the correct $\{Q, Q\}$ bracket. In the final section we consider "Cayley-Dickson halving" the octonionic M-algebra, which corresponds to its reinterpretation as the maximal supergravity algebra in $D = 7, 5, 4$. For example, the M-algebra may be considered to be an octonionic rewriting of the $D = 4$, $\mathcal{N} = 8$

supersymmetry algebra; from this perspective the $D = 4$, $\mathcal{N} = 1$ algebra comes from a truncation $\mathbb{O} \to \mathbb{R}$.

2 The division algebras

A normed division algebra is an algebra $\mathbb{A}$ equipped with a positive-definite norm satisfying the condition

$$||xy|| = ||x||\,||y||. \tag{2.1}$$

Remarkably, there are only four such algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, with dimensions $n = 1, 2, 4$ and 8, respectively.

A division algebra element $x \in \mathbb{A}$ is written as the linear combination of n basis elements with real coefficients: $x = x_a e_a$, with $x_a \in \mathbb{R}$ and $a = 0, \cdots, (n-1)$. One basis element $e_0 = 1$ is real; the other $(n-1)$ basis elements e_i are imaginary:

$$e_0^2 = 1, \qquad e_i^2 = -1, \tag{2.2}$$

where $i = 1, \cdots, (n-1)$. In analogy with the complex case, we define a conjugation operation indicated by $*$, which changes the sign of the imaginary basis elements:

$$e_0{}^* = e_0, \qquad e_i{}^* = -e_i. \tag{2.3}$$

The multiplication rule for the basis elements of a division algebra is given by:

$$\begin{aligned} e_a e_b &= (\delta_{a0}\delta_{bc} + \delta_{0b}\delta_{ac} - \delta_{ab}\delta_{0c} + C_{abc})\, e_c \equiv \Gamma^a_{bc} e_c, \\ e^*_a e_b &= (\delta_{a0}\delta_{bc} - \delta_{0b}\delta_{ac} + \delta_{ab}\delta_{0c} - C_{abc})\, e_c \equiv \bar{\Gamma}^a_{bc} e_c, \end{aligned} \tag{2.4}$$

where we define the structure constants[1]

$$\begin{aligned} \Gamma^a_{bc} &= \delta_{a0}\delta_{bc} + \delta_{b0}\delta_{ac} - \delta_{ab}\delta_{c0} + C_{abc}, \\ \bar{\Gamma}^a_{bc} &= \delta_{a0}\delta_{bc} - \delta_{b0}\delta_{ac} + \delta_{ab}\delta_{c0} - C_{abc} \quad \Rightarrow \quad \Gamma^a_{bc} = \bar{\Gamma}^a_{cb}. \end{aligned} \tag{2.5}$$

The tensor C_{abc} is totally antisymmetric with $C_{0ab} = 0$, so it is identically zero for $\mathbb{A} = \mathbb{R}, \mathbb{C}$. For the quaternions C_{ijk} is simply the permutation symbol ε_{ijk}, while for the octonions the non-zero C_{ijk} are specified by the set of oriented lines of the Fano plane, see [16].

One of the most important properties of the division algebras is that they provide a representation of the SO(n) Clifford algebra. This is reflected in the structure constants, which satisfy

$$\begin{aligned} \Gamma^a\bar{\Gamma}^b + \Gamma^b\bar{\Gamma}^a &= 2\delta^{ab}\mathbb{1}, \\ \bar{\Gamma}^a\Gamma^b + \bar{\Gamma}^b\Gamma^a &= 2\delta^{ab}\mathbb{1}. \end{aligned} \tag{2.6}$$

In other words, we have the interpretation that multiplying a divison algebra element ψ by the basis element e_a has the effect of multiplying ψ's components by the gamma matrix $\bar{\Gamma}^a$:

$$e_a\psi = e_a e_b \psi_b = \Gamma^a_{bc} e_c \psi_b = e_c \bar{\Gamma}^a_{cb}\psi_b. \tag{2.7}$$

[1] The unusual choice of index structure is for later convenience - see equations (2.6) and (2.7).

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This property is essential for many of the applications of division algebras to physics, including that of this paper.

A natural inner product [16] on $\mathbb{A}$ is given by:

$$\langle x|y\rangle = \frac{1}{2}\left(x^*y + y^*x\right) = x_a y_a \quad \text{i.e.} \quad \langle e_a|e_b\rangle = \delta_{ab}. \tag{2.8}$$

This is just the canonical inner product on $\mathbb{R}^n$.

3 A new outer product

It is interesting to see what other linear operations on $\mathbb{R}^n$ look like when written in terms of the division-algebraic multiplication rule. This was explored in [17], but we take a different approach here. Consider the following general problem. Given some linear operator on $\mathbb{R}^n$ expressed as an $n \times n$ matrix M_{ab}, we would like to find an operator $\hat{M}$ on the division algebra $\mathbb{A}$ such that $\hat{M}$ has the effect of multiplying the components of $x = x_a e_a \in \mathbb{A}$ by M_{ab}:

$$\hat{M}x \equiv e_a M_{ab} x_b. \tag{3.1}$$

An explicit form for this operator can be found using the inner product above. First we rewrite

$$\begin{aligned} M_{ab} &= M_{cd}\langle e_a|e_c\rangle\langle e_b|e_d\rangle \\ &= \frac{1}{2}M_{cd}\big\langle e_a|e_c(e_d^* e_b) + e_c(e_b^* e_d)\big\rangle. \end{aligned} \tag{3.2}$$

Now it is clear that the operator

$$\hat{M} \equiv \frac{1}{2}M_{cd}\Big(e_c\big(e_d^* \,\cdot\,\big) + e_c\big((\cdot)^* e_d\big)\Big), \tag{3.3}$$

where a dot represents a slot for an octonion, has matrix elements

$$\langle e_a|\hat{M}e_b\rangle = M_{ab}. \tag{3.4}$$

This suggests that we write the outer product for division algebra elements using their multiplication rule, defining:

$$\begin{aligned} \times : \mathbb{A} \otimes \mathbb{A} &\to \mathrm{End}(\mathbb{A}) \\ e_a \otimes e_b &\mapsto e_a \times e_b \equiv \frac{1}{2}\Big(e_a\big(e_b^* \,\cdot\,\big) + e_a\big((\cdot)^* e_b\big)\Big). \end{aligned} \tag{3.5}$$

With the new product comes the power to rewrite any expression involving $n \times n$ matrices and n-dimensional vectors in terms of multiplication in the n-dimensional division algebra $\mathbb{A}$.

It is useful to note various equivalent ways of writing the outer product above:

$$\begin{aligned} e_a \times e_b &= \frac{1}{2}\Big(e_a\big(e_b^* \,\cdot\,\big) + e_a\big((\cdot)^* e_b\big)\Big) \\ &= \frac{1}{2}\Big(\big(\,\cdot\, e_b^*\big)e_a + \big(e_b(\cdot)^*\big)e_a\Big) \\ &= \frac{1}{2}\Big(e_a\big(e_b(\cdot)^*\big) + e_a\big(\,\cdot\, e_b^*\big)\Big) \\ &= \frac{1}{2}\Big(\big((\cdot)^* e_b\big)e_a + \big(e_b^* \,\cdot\,\big)e_a\Big). \end{aligned} \tag{3.6}$$

Due to the alternativity of the division algebras we also have

$$e_a\big(e_b^*\,\cdot\,\big)+e_a\big((\cdot)^*e_b\big)=\big(e_ae_b^*\big)(\cdot)+\big(e_a(\cdot)^*\big)e_b, \tag{3.7}$$

and similarly for the other four possibilities above.

4 Octonionic spinors in $D = 11$

In $D = 11$ the Majorana spinor may be written as a 32-component real column vector. However, if we consider $\mathbb{R}^{32}$ as the tensor product $\mathbb{R}^4 \otimes \mathbb{R}^8 \cong \mathbb{R}^4 \otimes \mathbb{O}$ then we can write this as a 4-component octonionic column vector

$$\lambda=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix},\qquad \lambda_\alpha\in\mathbb{O},\quad \alpha=1,2,3,4. \tag{4.1}$$

A natural set of generators $\{\gamma^M\} = \{\gamma^0, \gamma^{a+1}, \gamma^9, \gamma^{10}\}$, $M = 0, 1, \ldots, 10$ for the 4×4 octonionic Clifford algebra is then given by

$$\begin{aligned}\gamma^0&=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix},\qquad \gamma^{a+1}=\begin{pmatrix}0&0&0&e_a^*\\0&0&e_a&0\\0&e_a^*&0&0\\e_a&0&0&0\end{pmatrix},\\ \gamma^9&=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},\qquad \gamma^{10}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\end{aligned} \tag{4.2}$$

with $a = 0, 1, \ldots, 7$. These matrices satisfy

$$\gamma^M\gamma^N+\gamma^N\gamma^M=2\eta^{MN}\mathbb{1}, \tag{4.3}$$

and the infinitesimal Lorentz transformation of the spinor λ is

$$\delta\lambda=\frac{1}{4}\omega_{MN}\gamma^M(\gamma^N\lambda), \tag{4.4}$$

where $\omega_{MN} = -\omega_{NM}$. In general, the action of the rank r Clifford algebra element on λ can be written

$$\gamma^{[M_1}\Big(\gamma^{M_2}\Big(\ldots\Big(\gamma^{M_{r-1}}\Big(\gamma^{M_r]}\lambda\Big)\Big)\ldots\Big)\Big). \tag{4.5}$$

The positioning of the brackets in the above expression follows from repeated application of (2.7); non-associativity matters only for the imaginary gamma matrices γ^{i+1}, which provide a representation of the SO(7) Clifford algebra. If we define an operator $\hat{\gamma}^M$, whose action is left-multiplication by γ^M, then we can think of the rank r Clifford algebra element as the operator

$$\hat{\gamma}^{[M_1M_2\ldots M_r]}\equiv\hat{\gamma}^{[M_1}\hat{\gamma}^{M_2}\ldots\hat{\gamma}^{M_r]}, \tag{4.6}$$

where the operators $\hat{\gamma}^M$ must be composed as

$$\hat{\gamma}^M \hat{\gamma}^N \lambda = \gamma^M \left(\gamma^N \lambda\right) \neq \left(\gamma^M \gamma^N\right) \lambda. \tag{4.7}$$

This ensures that the action of $\hat{\gamma}^{[M_1 M_2 \ldots M_r]}$ on a spinor is given by (4.5), as required.

5 The octonionic M-algebra

The anti-commutator of two supercharges in the $D = 11$ supergravity theory is conventionally written as the 'M-algebra' [18, 19]

$$\{Q_{\bar{\alpha}}, Q_{\bar{\beta}}\} = \left(\gamma^M C\right)_{\bar{\alpha}\bar{\beta}} P_M + \left(\gamma^{MN} C\right)_{\bar{\alpha}\bar{\beta}} Z_{MN} + \left(\gamma^{MNPQR} C\right)_{\bar{\alpha}\bar{\beta}} Z_{MNPQR}, \tag{5.1}$$

where $\bar{\alpha}, \bar{\beta} = 1, \ldots, 32$, P_M is the generator of translations and Z_{MN} and Z_{MNPQR} are the brane charges. The charge conjugation matrix $C_{\bar{\alpha}\bar{\beta}}$ serves to lower an index on each of the gamma matrices.

The left-hand side is a symmetric 32×32 matrix with 528 components, while the terms on the right-hand side consist of the rank 1, 2 and 5 Clifford algebra elements, which form a basis for such symmetric matrices. In terms of SO$(1, 10)$ representations:

$$(\mathbf{32} \times \mathbf{32})_{\text{Sym}} = \mathbf{11} + \mathbf{55} + \mathbf{462}. \tag{5.2}$$

We would like to write this algebra in terms of 4×4 octonionic matrices. However, the space of octonionic 4×4 matrices is of dimension $16 \times 8 = 128$, and hence naively does not carry nearly enough degrees of freedom to write (5.1).

The solution to this problem is to use the octonionic Clifford algebra operators $\hat{\gamma}^{[M_1 M_2 \ldots M_r]}$ defined in the previous section. These operators (including all ranks r) span a space of dimension $32 \times 32 = 1024$. In other words, their octonionic matrix elements are

$$\langle e_a | \hat{\gamma}^M{}_\alpha{}^\beta e_b \rangle = \gamma^M{}_{\alpha a}{}^{\beta b}, \quad \alpha, \beta = 1, 2, 3, 4, \tag{5.3}$$

and if we think of αa as a composite spinor index $\bar{\alpha} = 1, \ldots, 32$, then the set of $\{\gamma^M{}_{\bar{\alpha}}{}^{\bar{\beta}}\}$ generates the usual real Clifford algebra as in (5.1).

For the charge conjugation matrix, we define the 4×4 real matrix (which is numerically equal to γ^0 but with a different index structure)

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.4}$$

The octonionic matrix elements of this are then trivially

$$C_{\alpha a \beta b} = \langle e_a | C_{\alpha\beta} e_b \rangle = C_{\alpha\beta} \delta_{ab}, \tag{5.5}$$

which can be identified with the 32×32 matrix:

$$C_{\bar{\alpha}\bar{\beta}} = C_{\alpha a \beta b} = C_{\alpha\beta} \delta_{ab}. \tag{5.6}$$

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Armed with these tools, the right-hand side can then be written over $\mathbb{O}$ simply by replacing $\bar{\alpha} \to \alpha$ and putting hats on the gammas:

$$\left(\hat{\gamma}^M C\right)_{\alpha\beta} P_M + \left(\hat{\gamma}^{MN} C\right)_{\alpha\beta} Z_{MN} + \left(\hat{\gamma}^{MNPQR} C\right)_{\alpha\beta} Z_{MNPQR}. \tag{5.7}$$

With the identification $\bar{\alpha} = \alpha a$ we can also write the left-hand side of (5.1) in terms of the composite indices:

$$\{Q_{\bar{\alpha}}, Q_{\bar{\beta}}\} = \{Q_{\alpha a}, Q_{\beta b}\}. \tag{5.8}$$

Now, the expression (5.7) is an octonionic operator with matrix elements as on the right-hand side of (5.1), so on the left we require an octonionic operator

$$\widehat{\{Q_\alpha, Q_\beta\}} \tag{5.9}$$

with matrix elements given by (5.8). The required operator is obtained simply by contracting (5.8) with the outer product $e_a \times e_b$ defined in (3.5):

$$\widehat{\{Q_\alpha, Q_\beta\}} \equiv \{Q_{\alpha a}, Q_{\beta b}\} e_a \times e_b. \tag{5.10}$$

The octonionic formulation of the M-algebra is then

$$\widehat{\{Q_\alpha, Q_\beta\}} = \left(\hat{\gamma}^M C\right)_{\alpha\beta} P_M + \left(\hat{\gamma}^{MN} C\right)_{\alpha\beta} Z_{MN} + \left(\hat{\gamma}^{MNPQR} C\right)_{\alpha\beta} Z_{MNPQR}. \tag{5.11}$$

Using the first two versions of the outer product given in (3.6), we could write the left-hand side as

$$\widehat{\{Q_\alpha, Q_\beta\}} = \frac{1}{2}\Big(\left(Q_\alpha Q_\beta^*\right)(\cdot) + (\cdot)\left(Q_\beta^* Q_\alpha\right) + \left(Q_\alpha (\cdot)^*\right) Q_\beta + Q_\beta\left((\cdot)^* Q_\alpha\right)\Big). \tag{5.12}$$

The first two terms look similar to the more intuitive anti-commutator $\{Q_\alpha, Q_\beta^*\}$, explored in [15], but to reproduce the full M-algebra we require all four terms above.

6 Relation to lower dimensions

It is interesting to consider the octonionic version of the supersymmetry algebra after an $\mathbf{11} = \mathbf{4} + \mathbf{7}$ split:

$$\mathrm{SO}(1,10) \supset \mathrm{SO}(1,3) \times \mathrm{SO}(7). \tag{6.1}$$

Seven of the Clifford algebra generators γ^{i+1} are imaginary, while the other four are real. This suggests that we split the dimensions as follows:

$$\begin{aligned} M = 0, 1, \ldots, 10 \quad \to \quad & i + 1 = 1, \ldots, 8, \\ & \mu = 0, 1, 9, 10. \end{aligned} \tag{6.2}$$

In $D = 4$ we regard the $D = 11$ octonionic spinor $Q_{\alpha a} e_a$ as eight 4-component Majorana spinors $Q_{\alpha a}$, which we may leave packaged as an 'internal' octonion. This transforms as the spinor $\mathbf{8}$ of SO(7). The $D = 4$ interpretation of the octonionic gamma matrices is as follows:

$$\hat{\gamma}_{i+1} = \gamma_* \hat{e}_i, \tag{6.3}$$

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$D \setminus \mathcal{N}$	1	2	4	8
11	$\mathbb{O}^4$			
7	$\mathbb{H}^4$	$\mathbb{O}^4$		
5	$\mathbb{C}^4$	$\mathbb{H}^4$	$\mathbb{O}^4$	
4	$\mathbb{R}^4$	$\mathbb{C}^4$	$\mathbb{H}^4$	$\mathbb{O}^4$

Table 1. A summary of the division algebraic parameterisation of spinors used in $D = n + 3$ supersymmetry algebras. Note that supersymmetry algebras sharing the same $\mathbb{A}$ are equivalent and that Cayley-Dickson doubling $\mathbb{A}$ corresponds to doubling $\mathcal{N}$, or equivalently climbing upwards in dimension D.

where $\hat{e}_i$ denotes the operator whose action is left-multiplication by e_i and γ_* (otherwise known as γ_5) is the highest rank Clifford element:

$$\gamma_* = -\gamma^0\gamma^1\gamma^9\gamma^{10} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

The matrix $C_{\alpha\beta}$ is just the charge conjugation matrix in $D = 4$.

We do not split the M, N indices of equation (5.11) into μ and i parts here, as the expression of the right-hand side itself is not particularly illuminating. The result is a copy of the $\mathcal{N} = 8$ supersymmetry algebra written over the octonions. The interesting point is that the $D = 11$ supersymmetry algebra *can* be reinterpreted as an octonionic $D = 4$ algebra.

More generally, the spinor and associated gamma matrices defined in (4.1) and (4.2) correspond to those of $D = 4, 5, 7$ if we replace $\mathbb{O}$ with $\mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively — see table 1. This means that in this framework the minimal supersymmetry algebra in these dimensions is written over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, while doubling the amount of supersymmetry corresponds to Cayley-Dickson doubling the division algebra. This process terminates when we reach maximal supersymmetry, i.e. when the Cayley-Dickson process takes us to $\mathbb{O}$, the largest normed division algebra.

The above discussion serves to emphasise the correspondence between the octonions and maximal supersymmetry in various dimensions. Rather than thinking of the M-theory algebra as an eleven-dimensional real algebra, it may be fruitful to think of it as a four-dimensional octonionic one, as in table 2.

Acknowledgments

The work of LB is supported by an Imperial College Junior Research Fellowship. The work of MJD is supported by the STFC under rolling grant ST/G000743/1.

$D \setminus \mathcal{N}$	1	2	4	8
11	$\mathbb{R}^{32}$			
7	$\mathbb{R}^{16}$	$\mathbb{C}^{16}$		
5	$\mathbb{R}^{8}$	$\mathbb{C}^{8}$	$\mathbb{H}^{8}$	
4	$\mathbb{R}^{4}$	$\mathbb{C}^{4}$	$\mathbb{H}^{4}$	$\mathbb{O}^{4}$

Table 2. An alternative parameterisation of spinors used in $D = n + 3$ supersymmetry algebras. From this point of view the octonions single out $D = 4$.

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